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TECHNICAL NOTE 2058

ON THE CONTINUATION OF A POTENTIAL GAS FLOW ACROSS THE SONIC LINE

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SYMBOLS

The following notation is used in the ANALYSIS:

a	speed of sound
F_{α}	solution of hypergeometric equation (27)
K	coefficient in Chaplygin's equation, defined by equation (20)
K^*	step function approximating function K
K_*	function defined by equation (38)
M	Mach number
n	normal to the curve Γ pointing toward the domain Δ
q	speed
q_{cr}	critical speed
q_{max}	maximum value of q
s	arc length measured on Γ
S	length of Γ
t	function of speed (see equation (18))
t_{max}	value of t for $M = \infty$

u,v	components of velocity vector
x,y	coordinates in physical plane
$X(s),Y(s)$	coordinates of points on Γ
γ	exponent in pressure-density relation
Γ	curve in physical plane along which $M = 1$
Γ_h	image of Γ in hodograph plane
Δ	domain in which a subsonic flow is given
θ	angle between velocity vector and x-axis
ρ	density
ρ_{cr}	critical density
τ	variable defined by equation (28)
τ_{cr}	value of τ for $M = 1$
ϕ	velocity potential
ψ	stream function
ω	angle between tangent to Γ and x-axis
$\sim$	used over symbol, value on Γ
0	used as subscript, value for $q = 0$

The following notation is used in the appendix:

a,b	end points of segment along which Cauchy data are given
A,B,T,N,T',N',T",N"	constants defined in section 3 of the appendix
D(a,b)	domain defined in section 1 of the appendix
K	coefficient in equation (A1)
K_n	functions approximating function K
w,W	complex-valued functions
x,y	independent variables
$Y^{(n)},\tilde{Y}^{(n)}$	functions defined by equations (A18)
z,ζ	complex variables
$Z^n, i\cdot Z^{(n)}$	"formal powers" defined in section 4 of the appendix
Λ_ν	constants defined by equations (A34)
$\Sigma(K)$	matrix defined by equation (A9)
τ,ν	given initial values of ψ and its normal derivative
φ	function connected with ψ by equation (A7)
ψ	dependent variable in equation (A1)

ANALYSIS

1. The Continuation Theorem

The main result of this investigation reads as follows:

Given a domain Δ in the x,y-plane, whose boundary contains a continuously curved arc Γ (see fig. 1); given a subsonic potential gas flow in Δ; and if

(a) the Mach number of the flow on Γ is equal to unity,

(b) the components of the velocity vector and their first-order derivatives are continuous on Γ,

(c) the normal derivative of the speed on Γ is different from zero, and

(d) no streamline of the flow cuts Γ at a right angle; then the flow can be continued across any subarc of Γ as a potential supersonic flow without weak discontinuities and this continuation is uniquely determined.

This theorem contains no general statement concerning the extent of the supersonic region obtained; however, in each special case the method of continuation described as follows determines the extent.

By a compressible potential flow without weak discontinuities is meant a flow described by two twice continuously differentiable functions $\phi(x,y)$ and $\psi(x,y)$ (potential and stream function) satisfying the differential equations

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y} \\ \frac{\partial\phi}{\partial y} &= -\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial x} \end{aligned}\right\} \qquad (1)$$

where ρ_0 is the stagnation density and ρ the density given by

$$\rho = \rho_0\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{a_0{}^2}\right)^{\frac{1}{\gamma-1}} \tag{2}$$

Here γ is the exponent in the pressure—density relation; a_0 is the speed of sound at a stagnation point; and q, the speed, is

$$q = a_0\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2} \tag{3}$$

It is well known that in the supersonic region flows with weak discontinuities (discontinuities in the second derivatives of the velocity potential) are possible. In fact, there is no good physical reason to assume that in a supersonic flow the velocity components

$$\left.\begin{aligned} u &= a_0\frac{\partial\phi}{\partial x} \\ v &= a_0\frac{\partial\phi}{\partial y} \end{aligned}\right\} \tag{4}$$

possess partial derivatives. (The continuity equation and the condition for the existence of a potential may be expressed in terms of integral relations.) Under precisely which conditions flows with weak discontinuities can be obtained by continuing subsonic flows across the sonic line seems to be a very important but rather delicate question; it is intimately tied up with the question of the necessity of conditions (b) to (d).

If condition (c) is violated along an arc of Γ, then the velocity vector is constant along this arc. This follows from equation (8). It seems very probable that in such a case the velocity vector must be constant everywhere. Whether the continuation theorem remains true if condition (c) is violated at isolated points remains an open question.

Condition (d) is probably essential for the uniqueness of the continuation. If at some point P of Γ the streamline is normal to Γ, then the Mach lines at this point are tangential to Γ, for at Γ the Mach angle is 90°. Such a situation arises in a symmetrical Laval nozzle if the transition from subsonic to supersonic speeds occurs along a line extending across the nozzle (flow of Mayer type, see fig. 2). In a remarkable paper (reference 2) Frankl investigated this case and showed that the continuation of the subsonic flow into the supersonic region is not uniquely determined. He showed also that in this case the supersonic flow may exhibit weak discontinuities.

2. Reduction to a Cauchy Problem in the Hodograph Plane

In order to prove the theorem, let the equation of the curve Γ be

$$\left.\begin{aligned} x &= X(s) \\ y &= Y(s) \\ 0 &\leq s \leq S \end{aligned}\right\} \tag{5}$$

where s is the arc length of Γ measured in the direction of traversing Γ with the domain Δ to the left. By hypothesis $X''(s)$ and $Y''(s)$ exist and are continuous, and $X'^2 + Y'^2 = 1$. It follows that the angle

$$\omega(s) = \tan^{-1} \frac{Y'(s)}{X'(s)} \tag{6}$$

between the tangent to Γ at $x = X(s)$, $y = Y(s)$ and the positive x-direction is a continuously differentiable function of s. (See fig. 1.)

Set

$$u - iv = qe^{-i\theta} \tag{7}$$

The value of any function $F(x,y)$ at $x = X(s)$, $y = Y(s)$ will be denoted by $\tilde{F}(s)$. Thus $\tilde{\theta}(s) = \theta\left[X(s), Y(s)\right]$. Nikolskii and Taganov (reference 3) showed that

$$\tilde{\theta}'(s) = \sqrt{\frac{\gamma + 1}{2a_0^2}} \cos^2(\tilde{\theta} - \omega) \frac{\partial q}{\partial n} \tag{8}$$

where $\partial/\partial n$ denotes differentiation in the direction of the normal to Γ pointing toward the domain Δ. (This formula follows easily from the equation of motion written with q and θ as independent variables.) Since $M = 1$ on Γ and $M < 1$ in Δ ($M = q/a$ being the Mach number) it follows that $\partial q/\partial n \leq 0$, so that by hypotheses (c) and (d)

$$\tilde{\theta}'(s) < 0 \tag{9}$$

It is no loss of generality to assume that

$$2\pi > \tilde{\theta}(0) > \tilde{\theta}(S) > 0 \tag{10}$$

If this condition were not satisfied, Γ could be divided into a finite number of overlapping arcs along each of which θ would change by less than 2π. Inequalities (9) and (10) imply that Γ has a one-to-one image in the hodograph plane, which is the circular arc Γ_h given by

$$q = q_{cr}, \quad \tilde{\theta}(S) \leq \theta \leq \tilde{\theta}(0) \tag{11}$$

where q_{cr}, the critical speed, is

$$q_{cr} = a_0\sqrt{\frac{2}{\gamma + 1}}$$

By the definition of the potential and stream function,

$$\left.\begin{aligned} d\phi &= \frac{1}{a_0}\,(u\ dx + v\ dy) \\ d\psi &= \frac{\rho}{\rho_0 a_0}\,(-v\ dx + u\ dy) \end{aligned}\right\} \tag{12}$$

On Γ,

$$\tilde{u} = q_{cr} \cos \tilde{\theta}$$

$$\tilde{v} = q_{cr} \sin \tilde{\theta}$$

$$dx = ds \cos \omega$$

$$dy = ds \sin \omega$$

$$\rho = \rho_{cr} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}}$$

so that

$$\left.\begin{aligned} \tilde{\phi}(s) &= \left(\frac{2}{\gamma + 1}\right)^{1/2} \int_0^s \cos\,(\tilde{\theta} - \omega)\ ds \\ \tilde{\psi}(s) &= -\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \int_0^s \sin\,(\tilde{\theta} - \omega)\ ds \end{aligned}\right\} \tag{13}$$

These equations show that $\tilde{\phi}(s)$ and $\tilde{\psi}(s)$ are twice continuously differentiable. By hypothesis $\tilde{\theta}(s)$ is a continuously differentiable function. By virtue of inequality (9) the inverse function

$$s = s^*(\theta)$$

exists and hence is continuously differentiable. It follows that

$$\left.\begin{aligned} \phi^*(\theta) &= \tilde{\phi}\left[s^*(\theta)\right] \\ \tilde{\psi}^*(\theta) &= \tilde{\psi}\left[s^*(\theta)\right] \end{aligned}\right\} \tag{14}$$

are continuously differentiable functions of θ, $\tilde{\theta}(s) \leq \theta \leq \tilde{\theta}(0)$.

Assume now that the flow in Δ has been continued across Γ. Since

$$\frac{\partial(u,v)}{\partial(x,y)} = q\frac{\partial(q,\theta)}{\partial(x,y)}$$

it follows that on Γ

$$\frac{\partial(u,v)}{\partial(x,y)} = -\left(\frac{\partial q}{\partial n}\right)^2 \cos^2(\tilde{\theta} - \omega) \tag{15}$$

In fact, on Γ

$$\frac{\partial(q,\theta)}{\partial(x,y)} = \frac{\partial q}{\partial s}\frac{\partial \theta}{\partial n} - \frac{\partial q}{\partial n}\frac{\partial \theta}{\partial s}$$

and $\partial q/\partial s = 0$, whereas $\partial\theta/\partial s$ is given by equation (8). By hypotheses (c) and (d) expression (15) cannot vanish. Hence

$$\frac{\partial(u,v)}{\partial(x,y)} < 0$$

for all points sufficiently close to Γ. It follows that a sufficiently small neighborhood of any subarc of Γ has a one-to-one image in the hodograph plane, that is, a neighborhood U of a subarc of Γ_h. In the

hodograph plane the functions ϕ and ψ satisfy the well-known Chaplygin equations

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \theta} &= \frac{q\rho_0}{\rho} \frac{\partial \psi}{\partial q} \\ \frac{\partial \phi}{\partial q} &= -\frac{1 - M^2}{q} \frac{\rho_0}{\rho} \frac{\partial \psi}{\partial \theta} \end{aligned}\right\} \tag{16}$$

while on Γ_h they satisfy the initial conditions

$$\left.\begin{aligned} \phi(q_{cr},\theta) &= \phi^*(\theta) \\ \psi(q_{cr},\theta) &= \psi^*(\theta) \end{aligned}\right\} \tag{17}$$

for $\theta(S) < \theta < \theta(0)$.

3. Existence and Uniqueness of the Solution in the Hodograph Plane

In the preceding section the continuation of a subsonic flow across the sonic line has been reduced to the initial-value problem, equations (16) and (17). Introduce the new independent variable

$$t = \int_{q_{cr}}^{q} \frac{\rho \, dq}{q} \tag{18}$$

(Note that $q = 0$ corresponds to $t = -\infty$, $q = q_{cr}$ to $t = 0$, and $q = q_{max} = a_0 \sqrt{2/(\gamma - 1)}$ to some finite positive value of t.) In the θ,t-plane, ϕ and ψ satisfy the equations

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \theta} &= \frac{\partial \psi}{\partial t} \\ \frac{\partial \phi}{\partial t} &= -K(t)\frac{\partial \psi}{\partial \theta} \end{aligned}\right\} \tag{19}$$

where

$$K(t) = \frac{1 - M^2}{\rho^2} \tag{20}$$

The initial-value problem, equations (16) and (17), goes over into a Cauchy problem for equations (19). The theory of this Cauchy problem is given in the appendix. From the theorems stated in section 3 of the appendix it follows that the initial-value problem, equations (16) and (17), possesses a unique solution in the domain bounded by the arc Γ_h and two characteristics of equations (16). As is well known, these characteristics are epicycloids. In this domain (see fig. 3), which is called the "characteristic triangle determined by Γ_h," the functions ϕ and ψ will be continuously differentiable, provided that $\phi^*(\theta)$ and $\psi^*(\theta)$ have this property.

Remark.- Note that if ψ (considered as a function of θ and t) is twice continuously differentiable, then equations (19) imply that ψ satisfies Chaplygin's equation

$$K(t)\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{21}$$

In this case the initial-value problem considered may be formulated as a Cauchy problem for equation (21):

$$\psi(\theta,0) = \tau(\theta)$$

$$\psi_t(\theta,0) = \nu(\theta)$$

where

$$\tau(\theta) = \psi^*(\theta)$$

$$\nu(\theta) = \phi^{*\prime}(\theta)$$

In general, however, ψ need not be a twice continuously differentiable function in the supersonic part of the hodograph plane.

4. Existence and Uniqueness of the Solution in the Physical Plane

In order to complete the proof of the continuation theorem, it must be shown that the functions ϕ and ψ obtained in the preceding section can be transferred (in a unique way) to the physical plane (more precisely, to a part of the physical plane adjacent to Γ and lying outside Δ) and that in the x,y-plane these functions possess continuous derivatives of the second order which assume the "correct" values along Γ.

The functions ϕ and ψ obtained from the Cauchy problem coincide along Γ with the potential and stream function of the flow given in Δ and transferred to the hodograph plane by means of the mapping

$$\left.\begin{aligned} u &= u(x,y) \\ v &= v(x,y) \end{aligned}\right\} \tag{22}$$

It follows from equations (12) that the inverse mapping is given by

$$x + iy = a_0 \int \frac{e^{i\theta}}{q}\left(d\phi + i\frac{\rho_0}{\rho}\,d\psi\right) \tag{23}$$

or,

$$x + iy = a_0 \int \frac{e^{i\theta}}{q} \left[\left(\phi_\theta + i\frac{\rho_0}{\rho}\psi_\theta \right) d\theta + \left(\phi_q + i\frac{\rho_0}{\rho}\psi_q \right) dq \right]$$

Since the functions ϕ and ψ have been continued into the supersonic part of the hodograph plane, this integral is meaningful even there. Moreover, it is path-independent. Under the assumption that the partial derivatives $\phi_{\theta\theta}$, $\phi_{\theta q}$, . . . exist and are continuous, this follows immediately from Chaplygin's equations (16). But, in the case considered, this assumption cannot be made. Consider, however, the mapping from the hodograph plane to the ϕ,ψ-plane. By virtue of equations (16)

$$\frac{\partial(\phi,\psi)}{\partial(\theta,q)} = \frac{\rho}{\rho_0 q}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{\rho q}{\rho_0(1 - M^2)}\left(\frac{\partial\phi}{\partial q}\right)^2$$

It follows (for instance, from relations (9) and (13)) that this Jacobian is different from zero in some neighborhood of the arc Γ_h. Hence the mapping from the hodograph to the ϕ,ψ-plane is locally one-to-one, so that it is sufficient to establish the path-independence of the integral, equation (23), in the ϕ,ψ-plane. This leads at once to the conditions

$$\frac{\partial}{\partial\psi}\left(\frac{\cos\theta}{q}\right) = -\frac{\partial}{\partial\phi}\left(\frac{\rho_0 \sin\theta}{\rho q}\right)$$

$$\frac{\partial}{\partial\psi}\left(\frac{\sin\theta}{q}\right) = \frac{\partial}{\partial\phi}\left(\frac{\rho_0 \cos\theta}{\rho q}\right)$$

Using the relation

$$\frac{d}{dq}\left(\frac{\rho q}{\rho_0}\right) = 1 - M^2$$

these conditions may be transformed into

$$\left.\begin{aligned}\frac{\partial\theta}{\partial\phi} &= \frac{\rho}{\rho_0 q}\frac{\partial q}{\partial\psi}\\[2ex] \frac{\partial\theta}{\partial\psi} &= -\frac{\rho_0\left(1 - M^2\right)}{\rho q}\frac{\partial q}{\partial\phi}\end{aligned}\right\} \tag{24}$$

Interchanging the dependent and independent variables, it is seen that system (24) is equivalent to equations (16). Thus equations (16) insure the path-independence of the integral, equation (23), even without any assumption about the existence of second-order derivatives of ϕ and ψ.

Thus equation (23) defines a mapping also in the supersonic part of the hodograph plane. The Jacobian of this mapping is easily computed to be equal to

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{{a_0}^2}{q^4}\frac{\rho_0}{\rho}\left[\left(1 - M^2\right)\frac{{\rho_0}^2}{\rho^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\phi}{\partial\theta}\right)^2\right] \tag{25}$$

so that for $q = q_{cr}$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{{a_0}^2}\left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma-2}{\gamma-1}}\left[\phi^{*\prime}(\theta)\right]^2$$

But by equations (14)

$$\phi^{*\prime}(\theta) = \frac{d\tilde{\varphi}}{ds}\frac{ds^*}{d\theta} = \left(\frac{2}{\gamma + 1}\right)^{1/2}\frac{\cos(\tilde{\theta} - \omega)}{\tilde{\theta}'(s)} \neq 0$$

so that

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

on the arc defined by equation (11) and hence also in some neighborhood of any subarc of this arc. It follows that such a neighborhood is mapped onto a neighborhood of a subarc of Γ in a one-to-one manner. The functions ϕ and ψ obtained by solving the initial-value problem, equations (16) and (17), can now be transferred to the x,y-plane by means of the mapping defined by equation (23).

The derivatives of ϕ and ψ in the x,y-plane can be computed by using equations (16) and (23). (See reference 4.) This direct though lengthy computation yields exactly equations (12). Thus ϕ and ψ satisfy equations (1). Finally, equation (23) shows that x and y are continuously differentiable functions of u and v, so that u and v are continuously differentiable functions of x and y. Thus ϕ and ψ possess continuous partial derivatives of the second order in the x,y-plane. The proof is now complete.

5. Computation of the Solution

The theorems proved in the appendix implicitly contain methods for the effective computation of the solution. Thus theorem 2 shows that the solution of the Cauchy problem may be represented as in infinite series of particular solutions of the Chaplygin type. (The idea of the following proof is already contained in a remark by Frankl concerning the Tricomi problem; see reference 5.)

Chaplygin (reference 6) showed that a particular solution of the second-order equation obtained from equations (16) by eliminating ϕ can be found by setting (for any positive constant α)

$$\psi = e^{i\alpha\theta}(q/q_{max})^{\alpha}F_{\alpha}\left(q^2/q_{max}{}^2\right) \tag{26}$$

where q_{max} is the maximum speed possible for the given gas:

$$q_{max} = \sqrt{\frac{2}{\gamma - 1}}$$

and $F_{\alpha}(\tau)$ is any solution of the hypergeometric equation

$$\tau(1-\tau)F_\alpha''(\tau) + \left[1 + \alpha - \left(a_\alpha + b_\alpha + 1\right)\tau\right] F_\alpha'(\tau) - a_\alpha b_\alpha F_\alpha(\tau) = 0 \tag{27}$$

the constants a_α and b_α being determined by the equations

$$a_\alpha + b_\alpha = \alpha - \frac{1}{\gamma - 1}$$

$$a_\alpha b_\alpha = -\frac{\alpha(\alpha + 1)}{2(\gamma - 1)}$$

It is seen that the potential corresponding to equation (26) is given by

$$\phi = -ie^{i\alpha\theta}\left(\rho_0/\rho\right)\left(q/q_{max}\right)^\alpha \left[F_\alpha(\tau) - (2\tau/\alpha)F_\alpha'(\tau)\right]$$

where

$$\tau = \frac{q^2}{q_{max}^2} \tag{28}$$

Let $F_{\alpha I}(\tau)$ and $F_{\alpha II}(\tau)$ denote the particular integrals of equation (27) which for

$$\tau = \tau_{cr} = \frac{q_{cr}^2}{q_{max}^2}$$

satisfy the initial conditions

$$F_{\alpha I} = 0$$

$$F_{\alpha I}' = \frac{\alpha}{2}\left(\frac{2}{\gamma - 1}\right)^{\frac{1}{\gamma-1}}\left(\frac{\gamma + 1}{\gamma - 1}\right)^2$$

$$F_{\alpha II} = 1$$

$$F_{\alpha II}' = -\frac{\alpha}{2}\left(\frac{\gamma + 1}{\gamma - 1}\right)^2$$

Set

$$\left.\begin{aligned} B_{\alpha j} &= \tau^{\alpha/2} F_{\alpha j}(\tau) \\ A_{\alpha j} &= \frac{\rho_0}{\rho}\,\tau^{\alpha/2}\left[F_{\alpha j}(\tau) + \frac{2\tau}{\alpha} F_{\alpha j}'(\tau)\right] \\ & j = I,\ II \end{aligned}\right\} \tag{29}$$

then

$$\left.\begin{aligned} \phi_{\alpha j} &= e^{i\alpha\theta} A_{\alpha j}(\tau) \\ \psi_{\alpha j} &= i e^{i\alpha\theta} B_{\alpha j}(\tau) \end{aligned}\right\} \tag{30}$$

are pairs of solutions of equations (16) satisfying the initial conditions

$$\left.\begin{aligned} \phi_{\alpha j} &= 1, \psi_{\alpha j} = 0 \quad \text{for} \quad q = q_{cr}, \; j = \mathrm{I} \\ \phi_{\alpha j} &= 0, \psi_{\alpha j} = 1 \quad \text{for} \quad q = q_{cr}, \; j = \mathrm{II} \end{aligned}\right\} \tag{31}$$

Consider now the Cauchy problem, equations (16) and (17), assuming for the sake of simplicity that

$$\theta(S) = -T$$

$$\theta(0) = T$$

$$0 < T < \pi$$

Set

$$\alpha_n = \frac{n\pi}{T} \tag{32}$$

and expand the initial data $\phi^*(\theta)$ and $\psi^*(\theta)$ in Fourier series

$$\left.\begin{aligned} \phi^*(\theta) &= \sum_{n=0}^{\infty} a_n \cos \alpha_n\theta + b_n \sin \alpha_n\theta \\ \psi^*(\theta) &= \sum_{n=0}^{\infty} c_n \cos \alpha_n\theta + d_n \sin \alpha_n\theta \\ T &\leq \theta \leq T \end{aligned}\right\} \tag{33}$$

From the way the particular solutions, equations (30), have been constructed it follows that the series

$$\left.\begin{aligned}
\phi &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \alpha_n\theta + b_n \sin \alpha_n\theta\right) A_{\alpha_n I}(\tau) + \\
&\quad \sum_{n=1}^{\infty} \left(c_n \sin \alpha_n\theta - d_n \cos \alpha_n\theta\right) A_{\alpha_n II}(\tau) \\
\psi &= c_0 + \sum_{n=1}^{\infty} \left(c_n \cos \alpha_n\theta + d_n \sin \alpha_n\theta\right) B_{\alpha_n II}(\tau) + \\
&\quad \sum_{n=1}^{\infty} \left(-a_n \sin \alpha_n\theta + b_n \cos \alpha_n\theta\right) B_{\alpha_n I}(\tau)
\end{aligned}\right\} \tag{34}$$

represent a formal solution of the initial-value problem considered.

Theorem 2 of the appendix not only shows that the series converge absolutely and uniformly (within the "characteristic triangle") to the desired solution but also permits an estimate of the error committed by replacing the infinite series by finite partial sums. Theorems 4 and 5 of the appendix contain similar statements concerning the convergence of the differentiated series.

Remark 1.- While the theorems in the appendix refer to an equation of the form of equation (21), it is clear that these theorems imply similar ones for system (16). It may be noted that the convergence of series (34) was proved by a general method; no special properties of the hypergeometric functions were used.

Remark 2.- It should be noted that if the arc Γ_h on which the initial values are given exceeds

$$\pi\left(\sqrt{\frac{\gamma + 1}{\gamma - 1}} - 1\right)$$

then the epicycloids drawn from the extremities of the arc do not intersect. In this case the characteristic triangle must be replaced by the domain bounded by the initial arc, the two epicycloids, and an arc of the circle $q = q_{max}$. (See fig. 4.)

6. Computation of the Solution by a Second Method

The expansion in a Chaplygin series may not be the most efficient way of computing the solution of the initial-value problem in the hodograph plane, since it requires the preliminary computation of the functions A_α and B_α, and because of the possibility of slow convergence. Another method is suggested by theorems 3 and 6 of the appendix.

Let the equation satisfied by the stream function be transformed into the form of equation (23). Divide the interval $0 < t < t_{max}$ into subintervals

$$\left.\begin{aligned} & t_{\nu-1} < t < t_\nu \\ & \nu = 1, 2 \ldots n \\ & 0 = t_0 < t_1 < \ldots < t_n = t_{max} \end{aligned}\right\} \tag{35}$$

and set

$$K^*(t) = \min K(t') \tag{36}$$

$$t_{\nu-1} \leq t \leq t_\nu, \ t_{\nu-1} \leq t' \leq t_\nu$$

The equation

$$K^*(t)\psi_{\theta\theta} + \psi_{tt} = 0 \tag{37}$$

is an approximation to equation (21). Since $K^*(t)$ is a piecewise constant function (step function), it is equivalent to the one-dimensional wave equation in each strip $t_\nu < t < t_{\nu+1}$. A solution of equation (37) will be required to possess continuous partial derivatives of the first order. The potential ϕ corresponding to a given solution of equation (37) will be continuous but its first derivatives will possess discontinuities. The characteristics of equation (37) are given by the equations

$$\theta - \int \sqrt{-K^*(t)}\, dt = \text{Constant}$$

$$\theta + \int \sqrt{-K^*(t)}\, dt = \text{Constant}$$

which represent polygonal arcs in the θ,t-plane. (See figs. 5 and 6.)

For equation (37) the solution of the Cauchy problem, equations (24), is a trivial matter. This solution may be constructed by repeated use of the classical D'Alembert formula (as described in detail in section 6 of the appendix). Another possibility is to apply the method of the preceding section.

Theorems 3 and 6 assert that for a sufficiently fine subdivision, equations (35), the solution of equation (37) will be an arbitrarily close approximation to the solution of equations (24) with the same initial values.

Instead of using the approximating step function, equation (36), one may use the function

$$K_*(t) = \max K(t') \tag{38}$$

$$t_{\nu-1} \leq t \leq t_\nu,\ t_{\nu-1} \leq t' \leq t_\nu$$

In this case $K_*(t) \equiv 0$ for $0 < t < t_1$, so that in this strip ψ is a linear function of t.

In some cases it might be preferable to use as the approximating coefficient a piecewise linear function. In this case the approximating equation will be of the type of the Darboux-Tricomi equation

$$y\psi_{xx} + \psi_{yy} = 0 \tag{39}$$

in each strip $t_\nu < t < t_{\nu+1}$. The solution of the Cauchy problem for the Darboux-Tricomi equation can be expressed either by integral formulas (see reference 7) or by infinite series involving Bessel functions.

7. Application to Transonic Flows past Obstacles

The theorem stated in section 1 is of a purely local character. While in the hodograph plane the flow can be continued throughout the whole characteristic triangle, the transition to the physical plane may be possible only in the immediate neighborhood of the sonic line Γ. This may be due either to the existence of limiting lines or simply to the fact that a simply covered domain in the hodograph plane corresponds to a multiply covered domain in the physical plane.

In some important cases, however, it can be asserted that the continuation of the subsonic flow by the method described in sections 2 to 5 yields the whole supersonic region which is of relevance to the problem considered.

Consider a transonic flow past a closed body with a subsonic stream Mach number (and without shock waves). The supersonic regions S_1 and S_2 (see fig. 7) are bounded by solid walls and transition lines along which $M = 1$. For a supersonic region thus bounded, Nikolskii and Taganov (reference 3) proved an important theorem stating that the region S possesses a one-to-one image in the hodograph plane.

In this proof, Nikolskii and Taganov make use of the remark that all Mach lines in S must possess points in common with the transition line (the line $M = 1$). While they give no formal proof of this fact, a proof can be supplied without difficulty. The domain S is simply covered by a family of smooth curves, the streamlines of the flow, and the solid wall bounding S belongs to this family. A Mach line having no points in common with the transition line would have to originate and terminate at the solid wall. A simple argument, essentially equivalent to Rolle's theorem, shows that such a Mach line would be tangent to a streamline at at least one point P. But this is impossible since at P the Mach angle would have to be 0, which corresponds to $M = \infty$, $\rho = 0$.

Consider now the hodograph image S_h of S. It is bounded by the arc Γ_h of the circle $q = q_{cr}$ (corresponding to the transition line in the physical plane) and the curve W_h (the hodograph image of the solid boundary). Assume that S_h is not contained within the characteristic triangle determined by Γ_h which is bounded by Γ_h and two characteristics of the hodograph equations (i.e., two Busemann epicycloids). Then at least one of these Busemann epicycloids intersects W_h at two points, and S_h contains an arc of an epicycloid which has no points in common with Γ_h. This, however, is

impossible since the epicycloids are the hodograph images of the Mach lines. Thus S_h is contained within the characteristic triangle. By virtue of the previous results, this implies the following uniqueness theorem:

> Two transonic (subsonic at infinity) flows past an obstacle are identical if they are identical in the subsonic region.

Similar theorems are true for flows past curved walls and for Taylor type flows in nozzles (figs. 8 and 9). In the case of a Mayer type nozzle flow, however, the subsonic flow does not determine the supersonic flow uniquely.

The preceding remarks are of interest in connection with various attempts to construct transonic flows by the "correspondence method."

The correspondence method was first used by Chaplygin for jet problems (where, however, it is identical with the solution of the so-called direct problem). The correspondence method may be expressed in various analytical forms; however, except for the very original method of Bergman (reference 8) the basic procedure is always the same, as is clearly pointed out by Gelbart (reference 9). It consists of associating with a solution of the Cauchy-Riemann equations (representing an incompressible flow) a solution of Chaplygin's hodograph equations. In some cases the resulting compressible flow is of the same general character as the initial incompressible flow. To date, the most detailed application of this method to flows past obstacles is that given by Tsien and Kuo (reference 10). The continuation of a subsonic flow across the sonic line invariably occurs in the application of the correspondence methods to transonic flows.

While the mathematical problems connected with the correspondence method are as yet largely unsolved, the results of the present investigation show that once the solution of Chaplygin's equation in the subsonic part of the hodograph plane is determined, the continuation of the solution in the supersonic region is uniquely determined and can be effectively computed. If this continuation does not yield the desired result in the physical plane, the choice of the solution in the subsonic domain must be abandoned.

CONCLUDING REMARKS

The continuation of a given subsonic flow into the supersonic region has been discussed. The corresponding problem in the subsonic region is of little interest, since in the subsonic region the differential equations are of elliptic type with analytic coefficients

and have analytic solutions. Thus if the velocity potential ϕ is known up to a line Γ, the continuation of ϕ across Γ is a problem in analytic continuation. It seems improbable that any interesting and general statements can be made concerning this case. In the supersonic region the equations are of hyperbolic type and the continuation of a given flow across a line Γ reduces to the standard Cauchy problem. The theory of this problem is known (see reference 11, p. 326, and the literature quoted therein). It would be very interesting, however, to obtain results concerning a problem converse to the one considered here; that is, assume that a supersonic flow is known up to the line Γ on which the Mach number is equal to 1, and determine under which conditions the flow can be continued into the subsonic (elliptic) region.

Syracuse University
Syracuse, N. Y., September 1, 1947

APPENDIX

THE CAUCHY PROBLEM FOR CHAPLYGIN'S EQUATION

This appendix contains the proofs of the theorems used to establish the preceding results. For the convenience of the reader the notation standard in the theory of partial differential equations is used. Note that x and y no longer refer to the physical plane. Equation (A1) is essentially the same as equation (21).

1. Introduction

The aim of this investigation is to integrate the differential equation

$$K(y)\psi_{xx} + \psi_{yy} = 0 \tag{A1}$$

where

$$\left.\begin{aligned} K(0) &= 0 \\ K(y) &< 0 \\ \text{for} \quad y &< 0 \end{aligned}\right\} \tag{A2}$$

under the initial conditions

$$\left.\begin{aligned} \psi(x,0) &= \tau(x) \\ \psi_y(x,0) &= \nu(x) \\ a \leq x &\leq b \end{aligned}\right\} \tag{A3}$$

τ and ν being given functions. The solution is required in the domain $D(a,b)$ consisting of the segment $a \leq x \leq b$ of the axis $y = 0$ and of the part of the half plane $y < 0$ bounded by two characteristics of equation (A1) passing through the points $(a,0)$ and $(b,0)$.

Chaplygin's equation for the stream function as a function of the hodograph variables can be brought into the form of equation (A1), the domain $y < 0$ corresponding to the supersonic region. In the following considerations, however, no use will be made of the special form of the function K.

The classical theory fails to treat the Cauchy problem, equations (A1) to (A3), except for analytic τ and ν, because the initial data are given along the line $y = 0$ where the equation ceases to be of hyperbolic type. For one special case, that of the Darboux-Tricomi equation

$$y\psi_{xx} + \psi_{yy} = 0 \tag{A4}$$

the solution has been known for decades. In a recent paper, unavailable in this country, Christianovitch treated the case of the Chaplygin equation. According to Frankl (reference 1) he had to assume $\tau(x)$ and $\nu(x)$ to be analytic and had to use all their derivatives in order to obtain the solution. Frankl (reference 1) considered the case of nonanalytical Cauchy data for the equation

$$y\psi_{xx} + \psi_{yy} + a(x,y)\psi_x + b(x,y)\psi_y + c(x,y)\psi = 0 \tag{A5}$$

assuming the coefficients a, b, and c to be analytic. This treatment is based on special properties of the Riemann function for the Darboux-Tricomi equation and of the hypergeometric function and does not seem to lend itself easily to numerical computations. Note that if $K(y)$ is analytic and $K'(0) \neq 0$, equation (A1) can be brought to the form of equation (A5) by introducing

$$y' = \left(\frac{3}{2}\int_0^y \sqrt{f(y)}\, dy\right)^{2/3} \tag{A6}$$

as a new independent variable.

The present treatment, while limited to equations of the form of equation (A1), is of an entirely elementary character, requires only very simple regularity conditions for the function K, and remains valid even where a reduction to the form of equation (A5) is impossible (e.g., for $K = y^{1/3}$). Besides establishing the existence, uniqueness, continuity, and differentiability of the solution, it will be shown that the solution depends continuously on the Cauchy data (as noticed by Frankl), as well as on the coefficient K. These continuity properties imply two effective methods for the approximate computation of the solution: Either by representing it as a series of particular solutions of the Chaplygin type or by replacing the coefficient K by a piecewise constant function, in which case the determination of the solution becomes quite trivial. In fact, the approximation of $K(y)$ by step functions constitutes the main tool of this investigation.

2. Statement of the Problem

The function $K(y)$ will be assumed to satisfy the following conditions:

(1) $K(y)$ is defined and nondecreasing for $y \leq 0$

(2) $K(y)$ is continuous at $y = 0$

(3) $K(y)$ is negative, piecewise continuous, and possesses a piecewise continuous derivative for $y < 0$

Since the only interesting case is $K(0) = 0$, the condition that $K(y)$ be nondecreasing involves no serious loss of generality. (It is satisfied in the case of Chaplygin's equation.)

A function $\psi(x,y)$ will be called a regular solution of equation (A1) if ψ possesses continuous partial derivatives of the first order throughout its domain of definition, and piecewise continuous partial derivatives of the second order satisfying equation (A1). The Cauchy problem mentioned in the Introduction may be stated as follows:

> Problem A. Given two twice continuously differentiable functions $\tau(x)$ and $\nu(x)$ for $a \leq x \leq b$. To determine a regular solution $\psi(x,y)$ of equation (A1) defined in $D(a,b)$ and satisfying the initial conditions, equations (A3).

It will be convenient to consider instead a more general problem (suggested by gas dynamical applications). A regular solution ψ of equation (A1) determines (except for an additive constant) a function $\varphi(x,y)$ such that

$$\left.\begin{aligned} \varphi_x &= \psi_y \\ \varphi_y &= -K(y)\psi_x \end{aligned}\right\} \tag{A7}$$

(Note that the derivative φ_y might possess discontinuities.) The complex-valued function

$$\left.\begin{aligned} w(z) &= \varphi + i\psi \\ z &= x + iy \end{aligned}\right\} \tag{A8}$$

is called a Σ-monogenic function, where

$$\Sigma = \Sigma(K) = \begin{Vmatrix} 1 & 1 \\ 1 & K(y) \end{Vmatrix} \tag{A9}$$

denotes the coefficient-matrix of equations (A7). Equations (A7) imply that

$$\left.\begin{aligned} \int_\Gamma \varphi\, dx - K\psi\, dy &= 0 \\ \int_\Gamma \psi\, dx + \varphi\, dy &= 0 \end{aligned}\right\} \tag{A10}$$

for every closed rectifiable curve Γ contractible to a point. It will be convenient to call every continuous complex-valued function, equations (A8), satisfying equations (A10) a Σ-monogenic function.

Part of the theory of Σ-monogenic functions (references 12 and 13) (originally developed primarily in the elliptic case $K > 0$) remains

valid with this new definition. In particular, every Σ-monogenic function $w = \varphi + i\psi$ possesses a Σ-integral

$$W(z) = \int_{z_0}^{z} w(\zeta)\, d_\Sigma \zeta = \int_{z_0}^{z} \varphi\, dx - K\psi\, dy + i \int_{z_0}^{z} \psi\, dx + \varphi\, dy \tag{A11}$$

$W(z)$ is again a Σ-monogenic function, and

$$w = \frac{\partial W}{\partial x} \tag{A12}$$

Now, if ψ is a solution of problem A, then (for a proper normalization of φ)

$$\varphi(x,0) = \int_a^x \nu(\xi)\, d\xi, \quad a \leq x \leq b \tag{A13}$$

This remark suggests the following problem:

Problem B. Given two continuous functions $\tau(x)$ and $\nu(x)$ for $a \leq x \leq b$. To determine a Σ-monogenic function $w = \varphi + i\psi$ defined in $D(a,b)$ and satisfying the initial condition

$$w(x) = \int_a^x \nu(\xi)\, d\xi + i\tau(x), \quad a \leq x \leq b \tag{A14}$$

It is easy to see that if w is the solution of problem B, and if ψ possesses continuous first and piecewise continuous second derivatives, then ψ is a solution of problem A.

3. Statement of the Results

The main results of this investigation are:

Theorem 1 (Existence and uniqueness of the solution). Let $\tau(x)$ and $\nu(x)$ be arbitrarily given continuous functions

defined for $a \leq x \leq b$. There exists one and only one $\Sigma(K)$–monogenic function $w = \varphi + i\psi$ defined in $D(a,b)$ and satisfying the Cauchy condition

$$w(x) = \int_0^x \nu(\xi)\, d\xi + i\tau(x), \quad a \leq x \leq b \tag{A15}$$

<u>Theorem 2 (Continuous dependence on the initial values)</u>. The function $w = \varphi + i\psi$ defined in theorem 1 satisfies in $D(a,b)$ the inequalities

$$\left.\begin{aligned} |\varphi| &\leq AT + BN \\ |\psi| &\leq T + |y| N \end{aligned}\right\} \tag{A16}$$

where

$$T = \max |\tau(x)|$$

$$N = \max |\nu(x)|$$

$$a \leq x \leq b$$

$$A = A(y) = \sqrt{-K(y)}$$

$$B = B(x,y) = x - a + |y| A(y)$$

<u>Theorem 3 (Continuous dependence on the coefficient of the equation)</u>. Let $K_n(y)$ be a sequence of functions satisfying conditions (1) to (3). Let $w_n = \varphi_n + i\psi_n$ be the $\Sigma(K_n)$–monogenic function satisfying condition (A15), τ and ν being fixed continuous functions. If

$$\lim_{n\to\infty} K_n(y) = K(y)$$

uniformly for $0 \geq y \geq -H$, where H is the least upper bound of $|y|$ in $D(a,b)$, then

$$\lim_{n\to\infty} w_n(z) = w(z)$$

uniformly in $D(a,b)$, w being the $\Sigma(K)$-monogenic function satisfying condition (A15). (Cf. the remark made after lemma 4.)

Assuming that theorems 1 to 3 hold, it is easy to establish similar propositions for the derivatives of the function w.

Theorem 4. If $\tau(x)$ and $\nu(x)$ are continuously differentiable, then, using the notation of theorems 1 and 2, the partial derivatives φ_x, ψ_x, and ψ_y exist and are continuous in $D(a,b)$, whereas φ_y exists and is continuous in $D(a,b)$, except for those values of y for which $K(y)$ is discontinuous. Furthermore

$$|\varphi_x| = |\psi_y| \leq AT' + BN'$$

$$|\psi_x| \leq T' + |y| N'$$

$$|\varphi_y| \leq A^3T' + A^2BN'$$

where $T' = \max |\tau'|$ and $N' = \max |\nu'|$.

Theorem 5. If $\tau(x)$ and $\nu(x)$ possess continuous derivatives of the second order, then, using the notation of theorems 1, 2, and 4, the partial derivatives φ_{xx}, ψ_{xy}, and ψ_{xx} exist and are continuous in $D(a,b)$, whereas the derivatives φ_{xy}, φ_{yy}, and ψ_{yy} exist and are continuous in $D(a,b)$, except for those values of y for which $K(y)$ or $K'(y)$ is discontinuous. Furthermore

$$|\varphi_{xx}| = |\psi_{xy}| \leq AT'' + BN''$$

$$|\varphi_{xy}| = |\psi_{yy}| \leq A^2T'' + A^2|y|N''$$

$$|\varphi_{yy}| \leq |K'(y)| \ (AT' + |y|N') + A^3T'' + A^2BN''$$

where $T'' = \max |\tau''|$ and $N'' = \max |\nu''|$.

Theorem 6. Under the hypotheses of theorems 3 and 4 the partial derivatives of the first order of φ_n and ψ_n converge (uniformly in $D(a,b)$) toward those of φ and ψ. Under the hypotheses of theorems 3 and 5 the partial derivatives of the second order of φ_n and ψ_n (except perhaps $\varphi_{n,yy}$) converge (uniformly in $D(a,b)$) toward those of φ and ψ. The same is true for $\varphi_{n,yy}$ provided that $K'_n(y) \to K'(y)$ uniformly for $0 \geq y \geq -H$.

The proof of theorems 4 and 5 is based on the fact that if $w = \varphi + i\psi$ is a Σ-monogenic function, then the functions Φ and Ψ defined by

$$\Phi + i\Psi = \int w \ d_\Sigma z$$

possess continuous derivatives (except for possible discontinuities of Φ_y) satisfying equations (A7), so that if the functions $\tilde{\Phi}$ and $\tilde{\Psi}$ are defined by

$$\tilde{\Phi} + i\tilde{\Psi} = \int (\Phi + i\Psi) \ d_\Sigma z$$

then $\tilde{\Psi}$ is a regular solution of equation (A1).

Hence theorems 4 and 5 follow immediately from theorems 1 and 2 by applying the latter theorems to the Cauchy problems with the initial data $\tau'(x)$, $\nu'(x)$ and $\tau''(x)$, $\nu''(x)$. The proof of theorem 6 proceeds along similar lines and may be left to the reader.

It is clear that the functions and derivatives whose existence is asserted in theorems 1, 2, 4, and 5 exist and are continuous (or piecewise continuous) also along the characteristics bounding $D(a,b)$. In fact, the functions $\tau(x)$ and $\nu(x)$ and their derivatives have been assumed to be continuous in the closed interval (a,b), so that in order to establish the continuity of φ, ψ, ψ_x, and so forth in the closure of $D(a,b)$ it is necessary only to continue τ and ν in an appropriate manner over the interval $(a - \epsilon, \; b + \epsilon)$, $\epsilon > 0$, and apply the theorems to the domain $D(a - \epsilon, \; b + \epsilon)$. Theorems 1 to 6, however, remain valid if τ and ν and their derivatives are merely assumed to be continuous and bounded in the open interval (a,b).

It is rather obvious what modifications would be necessary if τ were assumed to be piecewise continuous. The function ν may be assumed to be bounded and measurable without endangering the validity of theorems 1 to 3, as will be seen from the proof.

4. Formal Powers

A significant class of Σ-monogenic functions is obtained by setting

$$Z^{(0)}(z) = 1$$

$$i \cdot Z^{(0)}(z) = i$$

and, for $n = 1, 2, \ldots,$

$$Z^{(n)}(z) = n \int_0^z Z^{(n-1)}(\zeta) \; d_\Sigma \zeta$$

$$i \cdot Z^{(n)}(z) = n \int_0^z i \cdot Z^{(n-1)}(\zeta) \; d_\Sigma \zeta$$

These "formal powers" admit the representation (see reference 12)

$$\left.\begin{aligned} Z^{(n)}(z) &= \sum_{\nu=0}^{n} \binom{n}{\nu} x^{n-\nu} i^{\nu} Y^{(\nu)}(y) \\ i\cdot Z^{(n)}(z) &= i \sum_{\nu=0}^{n} \binom{n}{\nu} x^{n-\nu} i^{\nu} \tilde{Y}^{(\nu)}(y) \end{aligned}\right\} \tag{A17}$$

where

$$Y^{(0)}(y) = \tilde{Y}^{(0)}(y) = 1$$

and, for $n = 1, 2, \ldots,$

$$\left.\begin{aligned} Y^{(2n+1)}(y) &= (2n + 1) \int_0^y Y^{(2n)}(\eta)\, d\eta \\ Y^{(2n)}(y) &= 2n \int_0^y K(\eta) Y^{(2n-1)}(\eta)\, d\eta \\ \tilde{Y}^{(2n+1)}(y) &= (2n + 1) \int_0^y K(\eta) \tilde{Y}^{(2n)}(\eta)\, d\eta \\ \tilde{Y}^{(2n)}(y) &= 2n \int_0^y \tilde{Y}^{(2n-1)}(\eta)\, d\eta \end{aligned}\right\} \tag{A18}$$

Note that the imaginary parts of $Z^{(n)}$ and $i\cdot Z^{(n)}$ possess continuous derivatives even if $K(y)$ possesses discontinuities.

It follows from the preceding formulas that (for a real x)

$$Z^{(n)}(x) = x^n$$

$$i \cdot Z^{(n)}(x) = ix^n$$

This implies

<u>Lemma 1</u>. If

$$\tau(x) = \sum_{j=0}^{N} a_j x^j$$

$$\nu(x) = \sum_{j=0}^{M} b_j x^j$$

then, for every real number a,

$$w(z) = \sum_{j=0}^{N} a_j i \cdot Z^{(j)}(z) + \sum_{j=0}^{M} \frac{b_j}{j+1}\left[Z^{(j+1)}(z) - Z^{(j+1)}(a)\right] \tag{A19}$$

is a Σ-monogenic function satisfying the initial condition

$$w(x) = \int_a^x \nu(\xi)\, d\xi + i\tau(x)$$

Formula (A19) is the analog of D'Alembert's solution of the equation of the vibrating string. In fact, if $K(y) = -\Lambda^2 =$ Constant, equation (A1) becomes

$$\psi_{xx} - \frac{1}{\Lambda^2}\psi_{yy} = 0$$

and a simple computation shows that in this case

$$Z^{(n)} = \frac{1}{2}\left[(x + \Lambda y)^n + (x - \Lambda y)^n\right] + \frac{i}{2\Lambda}\left[(x + \Lambda Y)^n - (x - \Lambda y)^n\right]$$

$$i \cdot Z^{(n)} = \frac{\Lambda}{2}\left[(x + \Lambda y)^n - (x - \Lambda y)^n\right] + \frac{i}{2}\left[(x + \Lambda y)^n + (x - \Lambda y)^n\right]$$

so that the imaginary part of equation (A19) is given by

$$\psi = \frac{\tau(x + \Lambda y) + \tau(x - \Lambda y)}{2} + \frac{1}{2\Lambda}\int_{x-\Lambda y}^{x+\Lambda y} \nu(\xi)\, d\xi$$

The analog of Bernoulli's solution is the one obtained by superimposing particular solutions of the Chaplygin type. (See section 5 of the ANALYSIS.)

Consider now two functions, $K_1(y)$ and $K_2(y)$, and denote functions (A18) formed with $K = K_i$ by $Y_i^{(n)}$ and $\tilde{Y}_i^{(n)}$ where $i = 1, 2$. Similarly, let the $\Sigma(K_i)$-monogenic formal powers be denoted by $Z_i^{(n)}$ and $i \cdot Z_i^{(n)}$.

Lemma 2. Set

$$\alpha = \alpha(H) = \max \left|K_i(y)\right|, \quad 0 \geq y \geq -H, \quad i = 1, 2$$

$$\delta = \delta(H) = \max \left|K_1(y) - K_2(y)\right|, \quad 0 \geq y \geq -H$$

There exist constants $M_n' = M_n'(\alpha)$ (depending only on α and n) such that

$$\left|Y_1^{(n)}(y) - Y_2^{(n)}(y)\right| \leq \delta M_n' |y|^n$$

$$\left|\tilde{Y}_1^{(n)}(y) - \tilde{Y}_2^{(n)}(y)\right| \leq \delta M_n' |y|^n$$

for $0 \geq y \geq -H$.

The proof follows easily by induction on n, using the definition of the function Y.

Lemma 3. There exist constants $M_n = M_n(\alpha)$ (depending only on α and n) such that

$$\left|Z_1^{(n)}(z) - Z_2^{(n)}(z)\right| \leq \delta M_n |z|^n$$

$$\left|i \cdot Z_1^{(n)}(z) - i \cdot Z_2^{(n)}(z)\right| \leq \delta M_n |z|^n$$

for $0 \geq y \geq -H$. Here α and δ have the same meaning as in lemma 2.

The proof follows immediately from lemma 2 and from equations (A17).

5. Main Lemma

The lemma to be established in this section accomplishes a twofold purpose. In the first place, it shows that theorem 3 holds whenever theorems 1 and 2 hold for the functions K_n and K. In the second place, it shows that, if theorems 1 and 2 hold for a sequence of functions K_n possessing a uniform limit K, then theorems 1 and 2 hold for K. Since every piecewise continuous function is a uniform limit of piecewise constant functions (step functions), it follows that theorems 1 to 3 are valid if theorems 1 and 2 hold for the case when K is a step function.

Lemma 4. Hypotheses: (a) For $n = 1, 2, \ldots$ the function $K_n(y)$ satisfies conditions (1) to (3). (b) For each $K_n(y)$

theorems 1 and 2 hold, as well as similar theorems concerning the Cauchy problem with initial data on the line $y = c < 0$.
(c) For $0 \geq y \geq -H$ the sequence $K_n(y)$ converges uniformly to a function $K(y)$ satisfying conditions (1) to (3).

Conclusion: Let $\tau(x)$ and $\nu(x)$ be continuous functions defined for $a \leq x \leq b$. It is assumed that $|y| \leq H$ in $D(a,b)$. Let $w_n(z)$ be a $\Sigma(K_n)$–monogenic function satisfying the condition

$$w_n(x) = \int_0^x \nu(\xi)\, d\xi + i\tau(x), \quad a \leq x \leq b \tag{A20}$$

Then, (a) in $D(a,b)$ the sequence w_n converges uniformly to a (continuous) function $w(z)$; (b) $w(z)$ is a $\Sigma(K)$–monogenic; (c) $w(z)$ satisfies the initial condition (A15); (d) $w(z)$ satisfies the inequalities (A16); and (e) $w(z)$ is the only function in $D(a,b)$ satisfying assertions (b) and (c).

Remark.– The functions w_n are defined, not in $D(a,b)$, but in domain $D_n(a,b)$ defined in the following proof. In view of equation (A22), one may speak of the convergence of the sequence w_n in $D(a,b)$, since each point of $D(a,b)$ belongs to all but a finite number of the regions $D_n(a,b)$.

Proof.– At first it will be assumed that assertion (a) holds. Set $w_n = \varphi_n + i\psi_n$ and $w = \varphi + i\psi$. For every closed curve Γ in $D(a,b)$ the relations

$$\left.\begin{aligned} \int_\Gamma \varphi\, dx - K\psi\, dy &= 0 \\ \int_\Gamma \psi\, dx + \varphi\, dy &= 0 \end{aligned}\right\} \tag{A21}$$

must hold in order that $w(z)$ be $\Sigma(K)$–monogenic. Let $D_n(a,b)$ denote the set defined with respect to $K_n(y)$ in the same way as $D(a,b)$ was defined with respect to $K(y)$. Hypothesis (c) implies that

$$\lim_{n \to \infty} D_n(a,b) = D(a,b) \tag{A22}$$

Hence, for sufficiently large values of n the curve Γ is in $D_n(a,b)$. Then

$$\left.\begin{aligned}\int_\Gamma \varphi_n\,dx - K_n\psi_n\,dy = 0\\ \int_\Gamma \psi_n\,dx + \varphi_n\,dy = 0\end{aligned}\right\} \qquad (A23)$$

so that equations (A21) follow from hypothesis (c) and assertion (a). Assertion (c) follows immediately from equation (A20). Assertion (d) is true, for the estimates, expressions (A16), hold for each function w_n according to hypothesis (a). In order to prove the uniform convergence of the sequence w_n, choose a positive ϵ and determine two polynomials

$$t(x) = \sum_{j=0}^{N} a_j x^j$$

$$n(x) = \sum_{j=0}^{M} b_j x^j$$

such that

$$\left.\begin{aligned}|t(x) - \tau(x)| < \epsilon\\ |n(x) - \nu(x)| < \epsilon\\ a \leq x \leq b\end{aligned}\right\} \qquad (A24)$$

(This is possible by virtue of Weierstrass's approximation theorem.) Let $W_n(z)$ be the $\Sigma(K_n)$–monogenic function satisfying the condition

$$W_n(x) = \int_0^x n(\xi)\, d\xi + it(x), \quad a \leq x \leq b \tag{A25}$$

Since theorem 2 is assumed to hold for $K = K_n$, it follows from conditions (A20), (A24), and (A25) that for all points in $D_n(a,b)$

$$\left|w_n(z) - W_n(z)\right| < L\epsilon, \quad n = 1, 2, \ldots \tag{A26}$$

where L is a constant independent of z, n, and ϵ. Denote by $Z_n^{(m)}$ and $i \cdot Z_n^{(m)}$ the $\Sigma(K_n)$–monogenic formal powers (formed with respect to the function K_n). Since W_n is uniquely determined by condition (A26), it follows from lemma 1 that

$$W_n(z) = \sum_{j=0}^{N} a_j Z_n^{(j)}(z) + \sum_{j=0}^{M} \frac{b_j}{j + 1}\left[i \cdot Z_n^{(j+1)}(z) - i \cdot Z_n^{(j+1)}(a)\right]$$

Hence lemma 3 and the uniform convergence of the sequence K_n imply the existence of a number M_ϵ such that for all values of z in $D(a,b)$

$$\left|W_{m+p}(z) - W_m(z)\right| < \epsilon, \quad m > M_\epsilon, \quad p = 1, 2, \ldots \tag{A27}$$

By virtue of expressions (A26) and (A27)

$$\left|w_{m+p}(z) - w_m(z)\right| < (2L + 1)\epsilon, \quad m > M_\epsilon, \quad p = 1, 2, \ldots$$

for all points of the intersection $D(a,b) \cap D_{m+p}(a,b) \cap D_m(a,b)$. Since L is fixed and ϵ arbitrary and since $D_n(a,b) \rightarrow D(a,b)$, assertion (a) follows.

It remains to prove assertion (e). It must be shown that, if $g(z)$ is $\Sigma(K)$–monogenic and $g(x) = 0$, $a \leq x \leq b$, g vanishes

identically in $D(a,b)$. It is no loss of generality to assume that $g(z)$ is continuous in the closure of $D(a,b)$ (since it would be sufficient to consider g in $D(a+\epsilon,\ b-\epsilon)$, $\epsilon > 0$). Then

$$\lim_{c \to 0} \max \left|g(x+ic)\right| = 0, \quad (x+ic)\epsilon D(a,b) \tag{A28}$$

Furthermore, no loss of generality is involved in assuming that the real and imaginary parts of $g(z)$ are continuously differentiable (for otherwise $g(z)$ could be replaced by its $\Sigma(K)$-integral). The argument leading to the proof of assertions (a) to (d) may be repeated to establish the existence of a $\Sigma(K)$-monogenic function $G_c(z)$ satisfying the condition

$$G_c(x+ic) = g(x+ic), \quad (x+ic)\epsilon D(a,b) \tag{A29}$$

and such that for $x + iy = z\epsilon D(a,b)$ and $y \leq c$

$$G_c(z) = \lim_{n\to\infty} G_{c,n}(z) \tag{A30}$$

$G_{c,n}(z)$ being a $\Sigma(K_n)$-monogenic function with

$$G_{c,n}(x+ic) = g(x+ic), \quad (x+ic)\epsilon D(a,b) \tag{A31}$$

Let $z_o = x_o + iy_o$ be any point of $D(a,b)$ with $y_o < c$. It follows from hypothesis (b) and theorem 2 that there exists a number L such that

$$\left|G_{c,n}(z_o)\right| \leq L \max \left|g(x+ic)\right|$$

so that by equation (A30)

$$\left|G_c(z_o)\right| \leq L \max \left|g(x+ic)\right|$$

and by equation (A28)

$$G_c(z_o) = 0$$

To show that $g \equiv 0$ it will suffice to show that $G_c(z) \equiv g(z)$ for all values of c. Set, c being fixed,

$$f(z) = g(z) - G_c(z)$$

and

$$F(z) = \Phi + i\Psi$$

$$= \int_{d+ic}^{z} d_{\Sigma(K)}\zeta \int_{d+ic}^{\zeta} f(\zeta') \, d_{\Sigma(K)}\zeta'$$

where d is such that $d + ic$ lies on the characteristic passing through $(a,0)$. The function $F(z)$ is $\Sigma(K)$–monogenic and

$$F(x + ic) = 0, \quad (x + ic)\epsilon D(a,b) \tag{A32}$$

The function Ψ possesses continuous derivatives Ψ_{xx} and Ψ_{xy} and an at least piecewise continuous derivative Ψ_{yy}. It satisfies equation (A1) and, in view of equation (A32), the initial condition

$$\Psi(x,c) = \Psi_y(x,c) = 0, \ (x,c)\epsilon D(a,b) \tag{A33}$$

But for $y < c$, equation (A1) is a hyperbolic type and the classical theory yields the result $\Psi \equiv 0$ in $D(a,b)$, for $y \leq c$. (The fact that K and K' are permitted to have discontinuities does not impair the validity of this argument.) Thus $\Phi \equiv 0$ and, since $f = \partial^2 F/\partial x^2$, $g \equiv G_c$. This completes the proof.

6. The Step-Function Case

In this section $K(y)$ is assumed to be a step function (cf. figs. 5 and 6; the latter shows the characteristics and the domain $D(a,b)$).

Set

$$K(y) = -\Lambda_\nu^2, \quad y_{\nu-1} \geq y \geq y_\nu \tag{A34}$$

where

$$\left.\begin{aligned} &0 = y_0 > y_1 > y_2 \ldots \\ &0 < \Lambda_1 < \Lambda_2 \ldots \end{aligned}\right\} \tag{A35}$$

Theorems 1 and 2 will be proved under this assumption. In view of the results of the preceding section, this insures the validity of theorems 1 to 3 in the general case.

Lemma 5. If K is given by equation (A34), a complex-valued continuous function $w = \varphi + i\psi$ is $\Sigma(K)$-monogenic if and only if it is continuous and admits the representation

$$w = \Lambda_\nu\left[f_\nu(x + \Lambda_\nu y) - g_\nu(x - \Lambda_\nu y)\right] + i\left[f_\nu(x + \Lambda_\nu y) + g_\nu(x - \Lambda_\nu y)\right], \quad y_{\nu+1} \geq y \geq y_\nu \tag{A36}$$

(Formulas connecting f_ν with $f_{\nu+1}$ are given in the following discussion.)

Proof.- It is easy to see that the condition is always sufficient and that it is necessary when ψ possesses continuous derivatives of the first order and piecewise continuous derivatives of the second order. Hence, if w is Σ-monogenic, the function

$$f(z) = \int^{z}\left[\int^{\zeta} w(\zeta') \, d_{\Sigma}\zeta\right] d_{\Sigma}z$$

admits a representation of the form of equation (A36). Since $w = f_{xx}$ the assertion follows.

Lemma 6. If K is given by equation (A34), theorem 1 holds.

Proof.— Set

$$w = \varphi + i\psi$$

$$\left.\begin{aligned} \varphi &= \Lambda_{\nu}\left[f_{\nu}(x + \Lambda_{\nu}y) - g_{\nu}(x - \Lambda_{\nu}y)\right] \\ \psi &= f_{\nu}(x + \Lambda_{\nu}y) + g_{\nu}(x - \Lambda_{\nu}y), \quad y_{\nu-1} \geq y \geq y_{\nu} \end{aligned}\right\} \tag{A37}$$

where

$$\left.\begin{aligned} f_{1}(\xi) &= \frac{1}{2}\tau(\xi) + \frac{1}{2\Lambda_{1}}\int_{a}^{\xi} \nu(\xi') \, d\xi' \\ g_{1}(\xi) &= \frac{1}{2}\tau(\xi) - \frac{1}{2\Lambda_{1}}\int_{a}^{\xi} \nu(\xi') \, d\xi' \end{aligned}\right\} \tag{A38}$$

and for $\nu = 1, 2, \ldots$

$$\left.\begin{aligned}
f_{\nu+1}(\xi) &= \frac{1}{2}\left(1 + \frac{\Lambda_\nu}{\Lambda_{\nu+1}}\right)f_\nu\left[\xi - (\Lambda_{\nu+1} - \Lambda_\nu)y_\nu\right] + \\
&\quad \frac{1}{2}\left(1 - \frac{\Lambda_\nu}{\Lambda_{\nu+1}}\right)g_\nu\left[\xi - (\Lambda_{\nu+1} + \Lambda_\nu)y_\nu\right] \\
g_{\nu+1}(\xi) &= \frac{1}{2}\left(1 - \frac{\Lambda_\nu}{\Lambda_{\nu+1}}\right)f_\nu\left[\xi + (\Lambda_{\nu+1} + \Lambda_\nu)y_\nu\right] + \\
&\quad \frac{1}{2}\left(1 + \frac{\Lambda_\nu}{\Lambda_{\nu+1}}\right)g_\nu\left[\xi + (\Lambda_{\nu+1} - \Lambda_\nu)y_\nu\right]
\end{aligned}\right\} \tag{A39}$$

It is easy to check that w is Σ-monogenic in $D(a,b)$ and that condition (A15) is satisfied. The uniqueness of this solution follows from lemma 5.

Lemma 7. If K is given by equation (A34), theorem 2 holds.

Proof.- Because of the linearity of the problem and because of the special form of inequalities (A16) it will suffice to prove these inequalities under the assumption that at least one of the functions τ and ν vanishes identically. Assume first that $\nu \equiv 0$. Using the representation of the solution given by equations (A37) to (A39), it is seen that

$$\max\left(|f_{\nu+1}|,|g_{\nu+1}|\right) \leq \max\left(|f_\nu|,|g_\nu|\right), \quad \nu = 1, 2, \ldots$$

and that

$$|f_1| = |g_1| \leq T/2$$

Hence

$$|f_{\nu+1}| \leq T/2$$

$$|g_{\nu+1}| \leq T/2$$

and by equations (A37)

$$|\varphi| \leq AT$$

$$|\psi| \leq T$$

since for K given by equation (34), $A(y) \geq \Lambda_\nu$ for $y \geq y_{\nu-1}$. Now suppose that $\tau \equiv 0$. In this case the solution admits the representation

$$w(z) = W_\nu(z) + \sum_{j=1}^{\nu-1} w_j(z), \quad y_{\nu-1} \geq y \geq y_\nu \tag{A40}$$

where

$$\left.\begin{aligned} W_\nu &= \Phi_\nu + i\Psi_\nu \\ w_j &= \varphi_j + i\psi_j \end{aligned}\right\} \tag{A41}$$

are Σ-monogenic functions determined by the initial conditions

$$\Phi_1(x,0) = \int_a^x \nu(\xi)\ d\xi$$

$$\Psi_1(x,0) \equiv 0$$

$$a \leq x \leq b$$

$$\Phi_{\nu+1}(x,y_\nu) = \Phi_\nu(x,y_\nu)$$

$$\Psi_\nu(x,y_\nu) \equiv 0$$

$$a_\nu \leq x \leq b_\nu,\ \nu = 1,\ 2,\ .\ .\ .$$

$$\varphi_\nu(x,y_\nu) = 0$$

$$\psi_\nu(x,y_\nu) = \Psi_\nu(x,y_\nu)$$

$$a_\nu \leq x \leq b_\nu,\ \nu = 1,\ 2,\ .\ .\ .$$

Here a_ν or b_ν denotes the abscissa of the point of intersection of the line $y = y_\nu$ with the left or right characteristic, respectively, bounding $D(a,b)$. Note that when K is given by equation (34) the characteristics of equation (A1) are polygonal paths having, for $y_{\nu-1} \geq y \geq y_\nu$, the slopes Λ_ν^{-1} and $-\Lambda_\nu^{-1}$. It is easy to verify that

$$a_1 - a = -\Lambda_1 y_1$$

$$a_\nu - a_{\nu-1} = \Lambda_\nu(y_{\nu-1} - y_\nu), \ \nu = 2, 3, \ldots \tag{A42}$$

For $0 \geq y \geq y_1$,

$$\Phi_1(x,y) = \frac{1}{2}\int_a^{x+\Lambda_1 y} \nu(\xi)\, d\xi + \frac{1}{2}\int_a^{x-\Lambda_1 y} \nu(\xi)\, d\xi$$

$$\Psi_1(x,y) = \frac{1}{2\Lambda_1}\int_{x-\Lambda_1 y}^{x+\Lambda_1 y} \nu(\xi)\, d\xi$$

Hence

$$\left|\Phi_1(x,y)\right| \leq (x - a)N \tag{A43}$$

so that, in particular,

$$\left|\Phi_1(a_1,y_1)\right| \leq (a_1 - a)N \tag{A44}$$

and

$$\left|\Phi_{1,x}(x,y_1)\right| \leq N \tag{A45}$$

Also

$$\left|\Psi_1(x,y)\right| \leq |y|\, N \tag{A46}$$

For $y_{\nu-1} \geq y \geq y_\nu$, $\nu = 2, 3, \ldots$

$$\Phi_{\nu+1}(x,y) = \Phi_\nu(a_\nu,y_\nu) + \frac{1}{2}\int_{a}^{x+\Lambda_{\nu+1}(y-y_\nu)} \Phi_{\nu,x}(\xi,y_\nu)\,d\xi +$$

$$\frac{1}{2}\int_{a_\nu}^{x-\Lambda_{\nu+1}(y-y_\nu)} \Phi_{\nu,x}(\xi,y_\nu)\,d\xi$$

$$\Psi_{\nu+1}(x,y) = \frac{1}{2\Lambda_\nu + 1}\int_{x-\Lambda_{\nu+1}(y-y_\nu)}^{x+\Lambda_{\nu+1}(y-y_\nu)} \Phi_{\nu,x}(\xi,y_\nu)\,d\xi$$

Hence

$$\left|\Phi_{\nu+1}(x,y)\right| \leq \left|\Phi_\nu\left(a_\nu,y_\nu\right)\right| + \left(x - a_\nu\right) \max \left|\Phi_{\nu,x}\left(\xi,y_\nu\right)\right| \tag{A47}$$

so that, in particular,

$$\left|\Phi_{\nu+1}(a_{\nu+1},y_{\nu+1})\right| \leq \left|\Phi_\nu(a_\nu,y_\nu)\right| + (a_{\nu+1} - a_\nu) \max \left|\Phi_{\nu,x}(\xi,y_\nu)\right| \tag{A48}$$

and

$$\left|\Phi_{\nu+1,x}(x,y_{\nu+1})\right| \leq \max \left|\Phi_{\nu,x}(\xi,y_\nu)\right| \tag{A49}$$

Also

$$\left|\Psi_{\nu+1}(x,y)\right| \leq \left|y - y_\nu\right| \max \left|\Phi_{\nu,x}(\xi,y_\nu)\right| \tag{A50}$$

From inequalities (A45) and (A49) it follows that

$$\left|\Phi_{\nu,x}(\xi,y_\nu)\right| \leq N,\ \nu = 1, 2, \ldots \tag{A51}$$

Using this and inequalities (A44) and (A48), it is seen that

$$\left|\Phi_\nu(a_\nu,y_\nu)\right| \leq (a_\nu - a)N$$

so that by inequalities (A43) and (A47)

$$\left|\Phi_\nu(x,y)\right| \leq (x - a)N,\ y_{\nu-1} \geq y \geq y_\nu,\ \nu = 1, 2, \ldots \tag{A52}$$

Also, by inequalities (A46), (A50), and (A51)

$$\left|\Psi_\nu(x,y)\right| \leq \left|y - y_{\nu-1}\right|N,\ y_{\nu-1} \geq y \geq y_\nu,\ \nu = 1, 2, \ldots \tag{A53}$$

so that in particular

$$\left|\Phi_\nu(x,y_\nu)\right| \leq \left|y_\nu - y_{\nu-1}\right|N,\ \nu = 1, 2, \ldots \tag{A54}$$

Using this last inequality and the previous result on the case $\nu \equiv 0$, it follows that

$$\left.\begin{aligned} \left|\varphi_j\right| &\leq A\left|y_j - y_{j-1}\right|N,\ j = 1, 2, \ldots \\ \left|\psi_j\right| &\leq \left|y_j - y_{j-1}\right|N,\ j = 1, 2, \ldots \end{aligned}\right\} \tag{A55}$$

Collecting inequalities (A52), (A53), and (A55), it is clear that

$$|\varphi(x,y)| \leq \left[(x - a) + |y|A(y)\right]N = B(x,y)N$$

$$|\psi(x,y)| \leq |y|N$$

This completes the proof.

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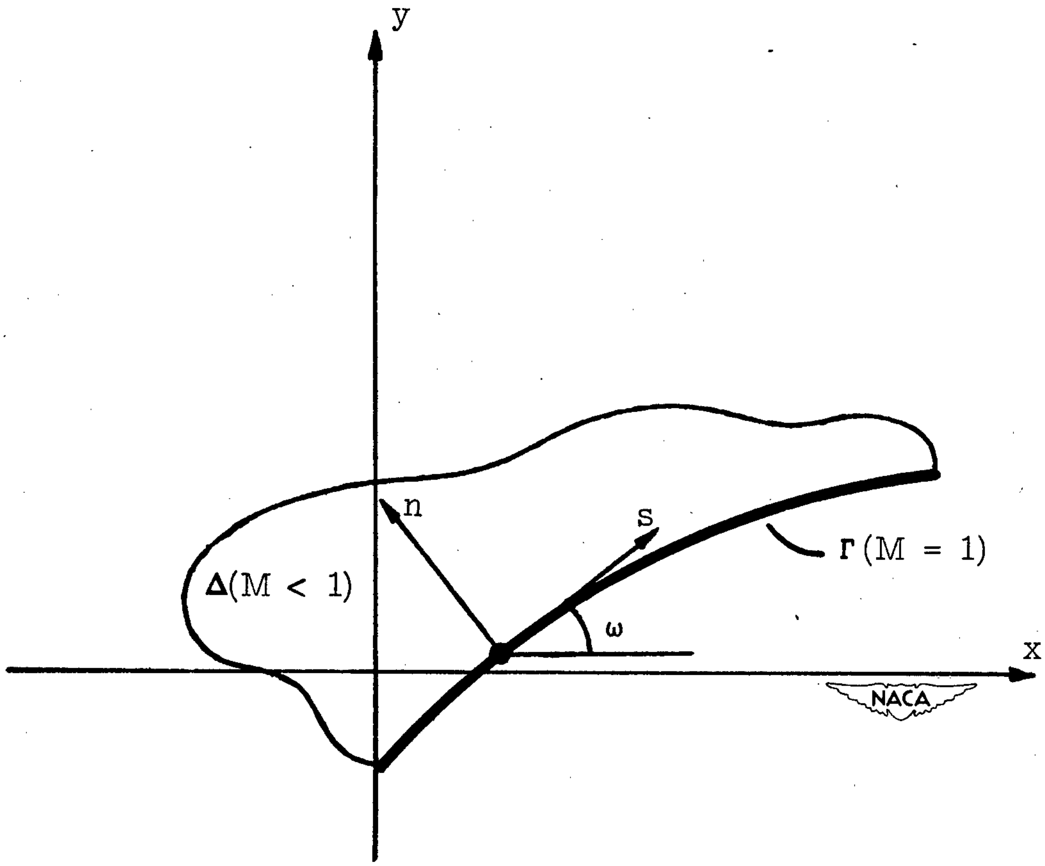

Figure 1.– Continuously curved arc Γ contained in boundary of domain Δ.

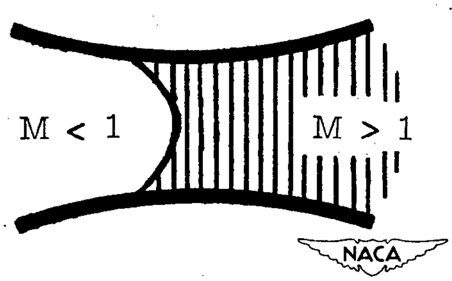

Figure 2.– Flow through Laval nozzle with transition from subsonic to supersonic flow occurring along a line extending across the nozzle.

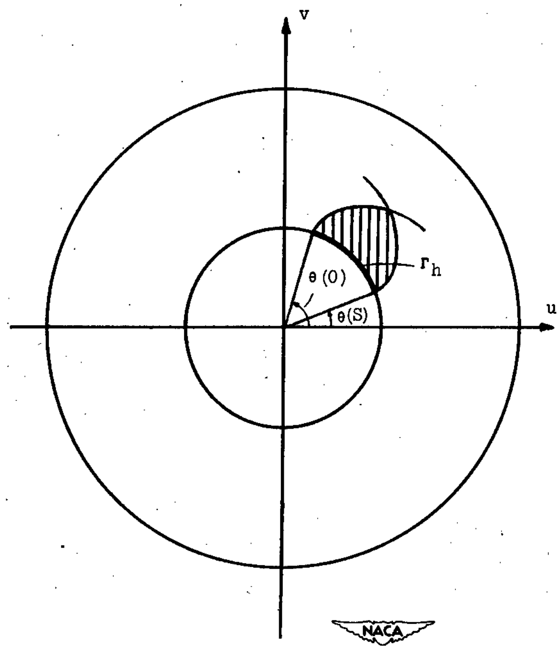

Figure 3.— Hodograph domain bounded by arc Γ_h and two epicycloids for the case of $\Gamma_h < \pi\left(\sqrt{\frac{\gamma + 1}{\gamma - 1}} - 1\right)$.

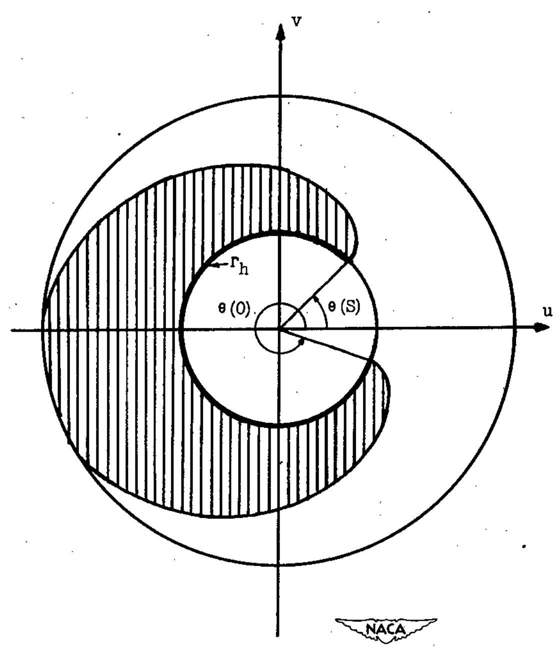

Figure 4.— Hodograph domain bounded by arc Γ_h, two epicycloids, and an arc of the circle $q = q_{max}$ for the case $\Gamma_h > \pi\left(\sqrt{\frac{\gamma + 1}{\gamma - 1}} - 1\right)$.

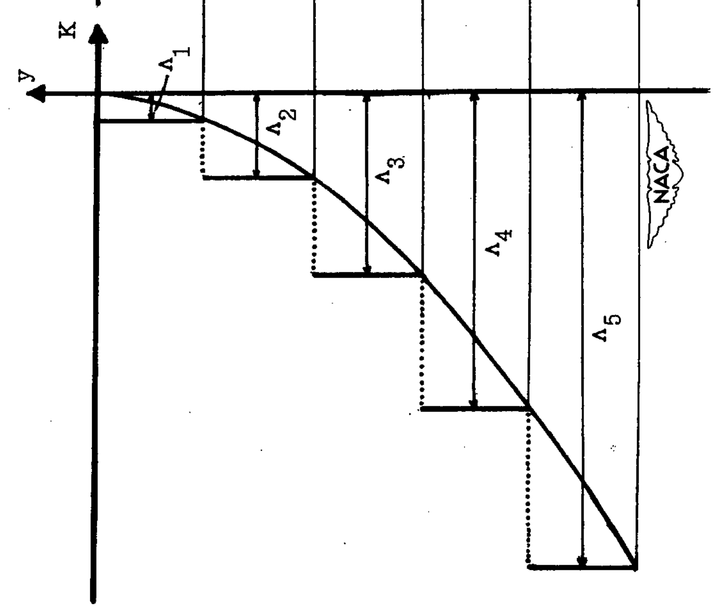

Figure 5.- Step function K(y).

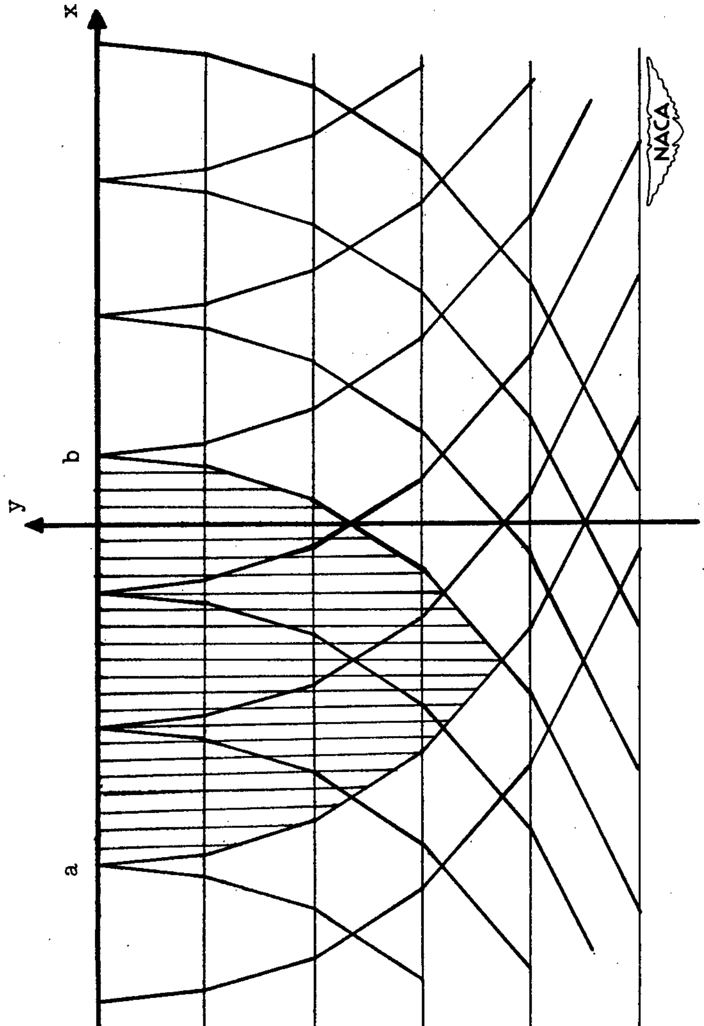

Figure 6.- Characteristics given by equation (37).

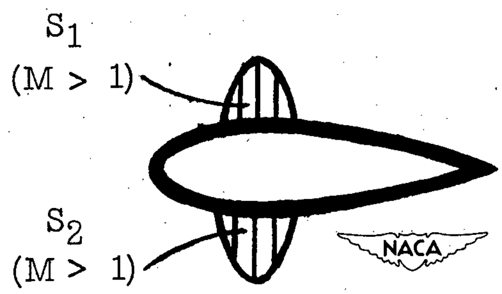

Figure 7.— Subsonic flow past a closed body with localized supersonic regions.

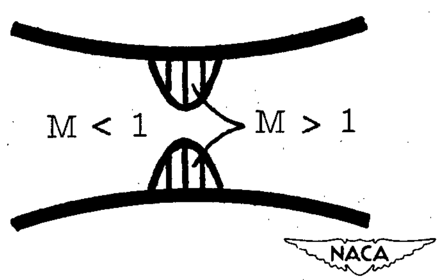

Figure 8.— Flow through Laval nozzle with localized supersonic regions.

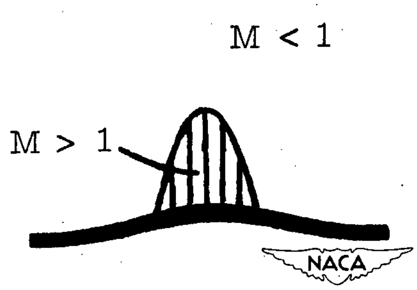

Figure 9.— Flow past a curved surface with a local supersonic region.